To: All

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Subject: Transient Conjugate Heat Transfer equations

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figures/tna_fluids_logo.png

This document introduces the set of partial differential equations describing the continuous model for *transient conjugate heat transfer* between a compressible viscous fluid and a solid.

Fluid simulation The behavior of the fluid is governed by the compressible Navier-Stokes equations. If we assume a problem with a single species, the set of PDEs for the fluid is:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) &= 0, \\
\frac{\partial \rho \mathbf{v}}{\partial t} + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) &= \operatorname{div}(-p\mathbb{I} + \boldsymbol{\tau}), \\
\frac{\partial \rho E}{\partial t} + \operatorname{div}(\rho \mathbf{v} E + p\mathbf{v}) &= \operatorname{div}(\boldsymbol{\tau}\mathbf{v}) - \operatorname{div}\mathbf{q},
\end{aligned}
\tag{1}
$$

where ρ represents the density of the fluid, $\mathbf{v}$ the velocity vector, p the pressure, E the specific total energy ($E + p/\rho$ represents the specific enthalpy), $\boldsymbol{\tau}$ is the viscous stress tensor, $\mathbf{q}$ is the heat flux and $\mathbb{I}$ represents the identity matrix. An equation of state (e.o.s) links the specific energy to the other variables - in the case of a calorically perfect gas for instance, the e.o.s reads:

$$
p = (\gamma - 1)\left(\rho E - \rho \frac{\mathbf{v}^2}{2}\right), \tag{2}
$$

where γ is the ratio of the isobaric to isochoric heat capacities of the fluid.

If we consider a Newtonian fluid following Stokes' hypothesis regarding the bulk dynamic viscosity, then the viscous stress tensor $\boldsymbol{\tau}$ is related to the velocity vector:

$$
\boldsymbol{\tau} = -\frac{2}{3}\mu(\operatorname{div}\mathbf{v})\,\mathbb{I} + \mu\left[\operatorname{grad}\mathbf{v} + (\operatorname{grad}\mathbf{v})^t\right], \tag{3}
$$

where μ is the dynamic viscosity of the fluid (predominantly dependent on the temperature).

The heat flux for a single species gas can be expressed using the Fourier's law of conduction:

$$
\mathbf{q} = -\kappa \operatorname{grad} T, \tag{4}
$$

where κ is the heat conductivity of the fluid (also predominantly dependent on the temperature).

For a two-dimensional problem, the set of equations (1) can be rewritten as follow for clarity:

$$
\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = 0, \tag{5}
$$

where

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{bmatrix}, \tag{6}$$

$$\mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p - \tau_{xx} \\ \rho u v - \tau_{yx} \\ (\rho E + p) u - \tau_{xx} u - \tau_{xy} v + q_x \end{bmatrix}, \tag{7}$$

$$\mathbf{G} = \begin{bmatrix} \rho v \\ \rho u v - \tau_{xy} \\ \rho v^2 + p - \tau_{yy} \\ (\rho E + p) v - \tau_{yx} u - \tau_{yy} v + q_y \end{bmatrix}, \tag{8}$$

$$\tau_{xx} = \frac{2}{3}\mu \left(2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right), \tag{9}$$

$$\tau_{yy} = \frac{2}{3}\mu \left(2\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right), \tag{10}$$

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \right), \tag{11}$$

$$q_x = -\kappa \frac{\partial T}{\partial x}, \tag{12}$$

and

$$q_y = -\kappa \frac{\partial T}{\partial y}. \tag{13}$$

Note that the aforementioned Navier-Stokes equations can be degenerated into the Euler equations by omitting the contributions of the viscous tensor $\boldsymbol{\tau}$. It can be a first step in building the numerical solver.

Solid simulation In the solid, the only transport mechanism that is considered here is the diffusion of heat. If we accept that the solid might be made of multiple materials whose properties might evolve with time (due to erosion, excessive heating, ...), then the PDE for the solid is:

$$\frac{\partial r C_p T}{\partial t} = \operatorname{div}\left(\lambda \operatorname{grad} T + \mathfrak{q}\right), \tag{14}$$

where r represents the density of the solid, C_p the specific heat capacity of the solid, λ the heat conductivity of the solid and $\mathfrak{q}$ stands for a spatially inhomogeneous heat source.

For a two-dimensional problem, and assuming there are no heat sources embedded in the solid, then equation (14) can be rewritten as follow for clarity:

$$\frac{\partial r C_p T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right). \tag{15}$$

As a first step in building the numerical solver for the conjugate heat transfer problem, the properties of the solid can be assumed to be spatially homogeneous and temporally constant such that equation (15) degenerates into:

$$\frac{\partial T}{\partial t} = \frac{\lambda}{r C_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right). \tag{16}$$

Fluid - Solid interface We assume that there is a perfect thermal contact between the fluid and the solid. Under this assumption, both normal heat fluxes and temperatures have to be continuous across the interface $\partial\Omega$, *i.e.* we want to impose that:

$$\begin{aligned} T_{\text{fluid}}|_{\partial\Omega} &= T_{\text{solid}}|_{\partial\Omega}\,, \\ \text{and} \quad \kappa\,\mathrm{grad}\,T_{\text{fluid}} \cdot \mathbf{n}|_{\partial\Omega} &= -\,\lambda\,\mathrm{grad}\,T_{\text{solid}} \cdot \mathbf{n}|_{\partial\Omega} \end{aligned} \tag{17}$$

where $\mathbf{n}|_{\partial\Omega}$ is the normal to $\partial\Omega$ pointing from the solid into the fluid, and the *fluid* and *solid* indices have been introduced for clarity to qualify respectively the fluid and the solid sides.