Derivation of a FV discretization for 1-D transient diffusion equations

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For a 1-D diffusion equation

$$\frac{du}{dt} = \frac{d}{dx}\left(\frac{du}{dx}\right) \tag{1}$$

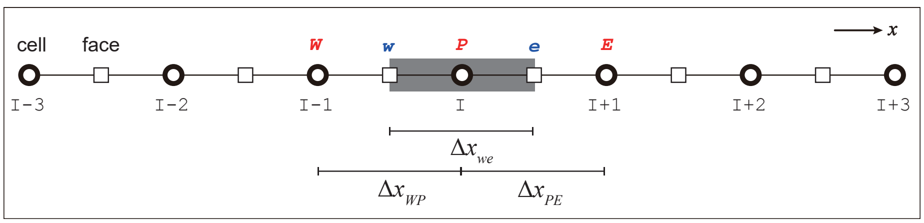

Figure 1:

Integrating both side of Equ. 1 over a FV cell, we have

$$\int_V \frac{du}{dt} dV = \int_V \frac{d}{dx}\left(\frac{du}{dx}\right) dV \tag{2}$$

where

$$\int_V \frac{du}{dt} dV \approx \Delta V \frac{du}{dt} = \Delta x \Delta y \frac{du}{dt} \tag{3}$$

and

$$\int_{\partial V} \left(\frac{du}{dx}\right) \cdot \mathbf{n} dS \approx \Delta y \left(\frac{du}{dx}\right)_e - \Delta y \left(\frac{du}{dx}\right)_w \tag{4}$$

Substituting Equ.3 and 4 into Equ.1 yields

$$\Delta x \Delta y \frac{du}{dt} = \Delta y \left(\frac{du}{dx}\right)_e - \Delta y \left(\frac{du}{dx}\right)_w \tag{5}$$

Divide both side of Equ.3 by $\Delta x \Delta y$, we have

$$\frac{du}{dt} \approx \frac{\left(\frac{du}{dx}\right)_e - \left(\frac{du}{dx}\right)_w}{\Delta x} \tag{6}$$

Approximating $\frac{\partial u}{\partial x}$ by central difference scheme yields

$$\begin{aligned}\left(\frac{du}{dx}\right)_e &\approx \frac{u_E - u_P}{\Delta x},\\ \left(\frac{du}{dx}\right)_w &\approx \frac{u_P - u_W}{\Delta x}\end{aligned} \tag{7}$$

and Equ.3 becomes

$$\frac{du}{dt} = \frac{\frac{u_E - u_P}{\Delta x} - \frac{u_P - u_W}{\Delta x}}{\Delta x} = \frac{u_E - 2u_P + u_W}{\Delta x^2} \tag{8}$$

or

$$\frac{du}{dt} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} \tag{9}$$

Equ.9 can be rewritten as

$$\frac{du}{dt} = G(t, u) \tag{10}$$

where $\mathbf{G}(t, u)$ is the RHS of Equ.9

$$\mathbf{G}(t, u) = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} \tag{11}$$

and is used by PETSc's **TS** object and implemented in the **TSSetRHSFunction()**.