3 Simplification

3.1 Unsteady, Incompressible

In non-dimensional form, the continuity equation, spread rate equation and pressure eigenvalue equation for 1D incompressible counter-flow are formulated as follows

$$\frac{d(\hat{\rho}\hat{u})}{d\hat{z}} + 2\hat{\rho}\hat{V} = 0, \tag{11}$$

$$\hat{\rho}\frac{d\hat{V}}{d\hat{t}} + \hat{\rho}\hat{u}\frac{d\hat{V}}{d\hat{z}} + \hat{\rho}\hat{V}^2 = -\hat{\Lambda}_r + \frac{1}{Re}\frac{d^2\hat{V}}{d\hat{z}^2}, \tag{12}$$

$$\hat{\Lambda}_r = const, \tag{13}$$

where $\hat{z} = z/L \in [0,1]$; $\hat{t} = U_{in}t/L$; $\hat{\rho} = \rho/\rho_{in} = 1$ in this incompressible case; $\hat{u} = u/U_{in}$; $\hat{V} = VL/U_{in}$; $\hat{\Lambda}_r = \Lambda_r L^2/\rho_{in}U_{in}^2$ and $Re = \rho_{in}U_{in}L/\mu$.

Boundary conditions are prescribed as follows

$$\hat{u}(0) = \hat{u}_L, \hat{u}(1) = \hat{u}_R, \tag{14}$$

and

$$\hat{V}(0) = \hat{V}(1) = 0. \tag{15}$$

These equations are expressed in DAE form, leading to the residual function defined as

$$F(t, Y, \dot{Y}) = 0. \tag{16}$$

With N uniformly distributed grid points, the solution vector is arranged as

$$Y = \{\hat{\Lambda}_r, \hat{u}_1, \hat{V}_1, ..., \hat{u}_i, \hat{V}_i, ..., \hat{u}_N, \hat{V}_N\}^T, \tag{17}$$

and the corresponding residual vector is defined as

$$F = \{R_{\hat{\Lambda}_r}, R_{\hat{u}_1}, R_{\hat{V}_1}, ..., R_{\hat{u}_i}, R_{\hat{V}_i}, ..., R_{\hat{u}_N}, R_{\hat{V}_N}\}^T. \tag{18}$$

For discretization, the convective terms are simply handled by first-order upwind scheme, while diffusive terms are treated by second-order central scheme. By neglecting $\hat{\rho}$, which equals to 1 in this case, the residuals are discretized as

$$R_{\hat{u}_i} = \frac{\hat{u}_i - \hat{u}_{i-1}}{\Delta\hat{z}} + 2\Big(\frac{\hat{V}_i + \hat{V}_{i-1}}{2}\Big), \quad (2 \le i \le N), \tag{19}$$

and

$$R_{\hat{V}_i} = \begin{cases} \dot{\hat{V}}_i + \hat{u}_i\frac{\hat{V}_i - \hat{V}_{i-1}}{\Delta\hat{z}} + \hat{V}_i^2 + \hat{\Lambda}_r - \frac{1}{Re}\frac{\hat{V}_{i-1} - 2\hat{V}_i + \hat{V}_{i+1}}{\Delta\hat{z}^2}, & \hat{u}_i > 0 \\ \dot{\hat{V}}_i + \hat{u}_i\frac{\hat{V}_{i+1} - \hat{V}_i}{\Delta\hat{z}} + \hat{V}_i^2 + \hat{\Lambda}_r - \frac{1}{Re}\frac{\hat{V}_{i-1} - 2\hat{V}_i + \hat{V}_{i+1}}{\Delta\hat{z}^2}, & \hat{u}_i < 0 \end{cases} \tag{20}$$

for $2 \le i \le N-1$.

Treatment of boundary conditions and the pressure eigenvalue equation deserves attention. Since the continuity equation is a first-order equation, only one boundary condition is required

$$R_{\hat{u}_1} = \hat{u}_1 - \hat{u}_L. \tag{21}$$

Notice that no boundary condition is prescribed for the pressure eigenvalue, $\hat{\Lambda}_r$ should be determined from other conditions so that its coupling and influence can be correctly embodied. This is accomplished by setting

$$R_{\hat{\Lambda}_r} = \hat{u}_N - \hat{u}_R. \tag{22}$$

Since the spread rate equation is a spatially second-order equation, two boundary condition are required. The corresponding zero-value conditions are applied as

$$R_{\hat{V}_1} = \hat{V}_1, \tag{23}$$

$$R_{\hat{V}_N} = \hat{V}_N. \tag{24}$$

In this way, the DAE is mathematically well-posed.

By default, the temporal derivative is handled by the backward-Euler scheme. The implicit solution procedure necessitates evaluation of the jacobian matrix, which is defined as

$$J = \frac{\partial F}{\partial Y}. \tag{25}$$

It obvious that the first diagonal element of J is analytically 0.