Poisson linear system

Lionel Cheng

2021

In these notes we study linear systems associated to the discretization on a mesh of a Poisson problem

$$\begin{cases} \nabla^2 \phi = -R & \text{in } \dot{\Omega} \\ \phi = \phi_D & \text{on } \partial\Omega_D \\ \nabla\phi \cdot \mathbf{n} = -E_n & \text{on } \partial\Omega_N \end{cases} \tag{1}$$

The associated linear system will be denoted $Ax = b$.

1 Regular mesh

1.1 Cartesian

In 1D cartesian mesh with $x_0 = 0$, $x_N = L$ we have for Dirichlet BCs

$$A = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 1 & -2 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & -2 & 1 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix} \tag{2}$$

The matrix can be symmetrized to yield

$$A = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & -2 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & -2 & 0 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix} \tag{3}$$

and the removed values will be added to b in this modified version. Let us take the matrix of size N which represents the inner part of the Dirichlet matrices:

$$A = \begin{bmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix} \tag{4}$$

This matrix has eigenvalues $\lambda_k = 2(1 - \cos(k\pi/(N+1)))$ and eigenvectors $(v_k)_i = \sin(ik\pi/(N+1))$

For Neumann at x_0 and Dirichlet at x_N two versions are possible depending on the discretization of the Neumann condition. A ghost-cell approach yields

$$A = \begin{bmatrix} -2 & 2 & \dots & \dots & 0 \\ 1 & -2 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & -2 & 1 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix} \tag{5}$$

whereas a first-order discretization gives

$$A = \begin{bmatrix} -1 & 1 & \dots & \dots & 0 \\ 1 & -2 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & -2 & 1 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix} \tag{6}$$

1.2 Cylindrical

We consider the 1D Poisson equation in cylindrical coordinates:

$$\nabla^2 \phi = \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(r \frac{\mathrm{d}\phi}{\mathrm{d}r} \right) \tag{7}$$

The domain goes from $r = 0$ to $r = R$ so that a Neumann boundary condition is mandatory and a Dirichlet BC is applied at the top. For the bottom node special treatment is needed and there is a factor 4.

For interior nodes:

$$\nabla^2 \phi|_j = \frac{1}{r_j \Delta r} (r_{j+1/2} \frac{\phi_{j+1} - \phi_j}{\Delta r} - r_{j-1/2} \frac{\phi_j - \phi_{j-1}}{\Delta r}) \tag{8}$$

$$\nabla^2 \phi|_j = \frac{1}{\Delta r^2} (\frac{r_{j+1/2}\phi_{j+1} + r_{j-1/2}\phi_{j-1}}{r_j} - 2\phi_j) \tag{9}$$

$$\tag{10}$$

And for a constant spaced mesh we have therefore

$$A = \begin{bmatrix} -4 & 4 & \dots & \dots & 0 \\ 1/2 & -2 & 3/2 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & (n-3/2)/n & -2 & (n-1/2)/n \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix} \tag{11}$$

What is the spectrum and eigenvectors of this matrix? The 1D case is very close to the analytical solution of the problem which yields the solution in terms of Fourier series (with sines or cosines depending on the type of boundary condition). The eigenvectors are just the discretized version of the sine modes. Could this also be true for the 1D cylindrical matrix? The solution is more involved as it inclues Fourier-Bessel series.

2 Unstructured mesh

The general formulation of the laplacian in the interior nodes is as follows. $\mathbf{S}_k$ is the nodal normal, $E(i)$ the neighboring cells of a given node i, τ indicates a cell. Integrating on the nodal volume:

$$\int_{V_i} \nabla^2 \phi dV = \sum_{\tau \in E(i)} \sum_{k \in \tau} - \frac{\mathbf{S}_k \cdot \mathbf{S}_i}{V_\tau n_d^2} \phi_k \tag{12}$$

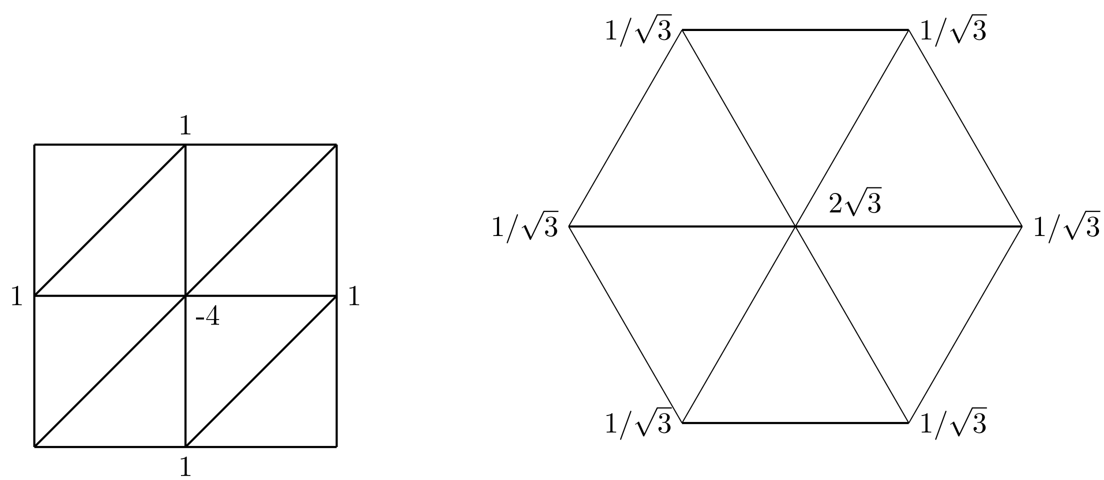
Each line is considered to be a node and the overall sum on the right would be its coefficients. In the end:

$$\sum_{\tau \in E(i)} \sum_{k \in \tau} \frac{\mathbf{S}_k \cdot \mathbf{S}_i}{V_\tau n_d^2} \phi_k = \frac{\rho_i V_i}{\varepsilon_0} \tag{13}$$

$$\sum_{\tau \in E(i)} \sum_{k \in \tau} A_{ik} \phi_k = \frac{\rho_i V_i}{\varepsilon_0} \tag{14}$$

$$\implies A\phi = b \tag{15}$$

Let us compare this formulation in 2D for two different kinds of triangles: triangles cutted from squares and equilateral triangles. The classical five point molecule is retrieved for the first case whereas a seven point molecule is retrieved for equilateral triangles.



(a) Triangular elements

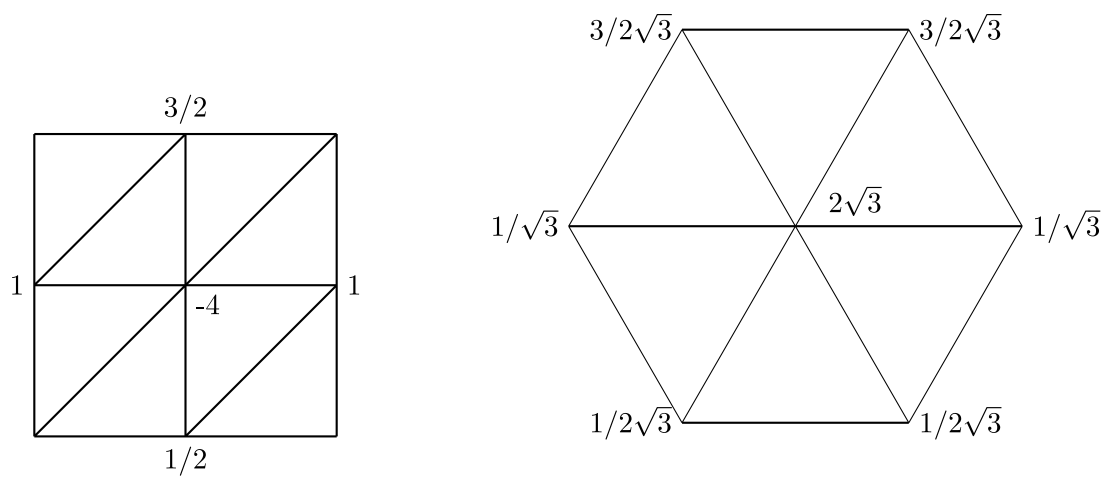
(b) Equilateral triangles

Figure 1: Laplacian cells.

For axisymmetric mesh in (x, r) the discretization is a bit more involved:

$$
\begin{aligned}
\nabla^2\phi \,\mathrm{d}V &= \int r\nabla^2\phi \,\mathrm{d}r \,\mathrm{d}x && (16)\\
&= \int_{A_i} \nabla_{2D} \cdot (r\nabla\phi) \,\mathrm{d}A && (17)\\
&= \int_{\partial A_i} r\nabla\phi \cdot \mathbf{n} \,\mathrm{d}l && (18)\\
&= \sum_{\tau\in E(i)} \int_{\partial A_i\cap\tau} r\nabla\phi \cdot \mathbf{n} \,\mathrm{d}l && (19)\\
&= \sum_{\tau\in E(i)} \nabla\phi_\tau \cdot \int_{\partial A_i\cap\tau} r\mathbf{n} \,\mathrm{d}l && (20)
\end{aligned}
$$

The matrices for the two kind of triangles studied above yield for the first cell at the axis the cells show in Fig. 2.



(a) Triangular elements at first node

(b) Equilateral triangles at first node

Figure 2: Laplacian cells in axysymmetric formulation.