

$$\begin{aligned}
F_1(p_w, S_n) &= \phi \frac{\partial(\rho_w(1 - S_n))}{\partial t} - \operatorname{div} \left( \rho_w \frac{k_{rw}(S_w)}{\mu_w} \mathbb{K} (\nabla p_w - \rho_w \mathbf{g}) \right) - q_w \\
F_2(p_w, S_n) &= \phi \frac{\partial(\rho_n S_n)}{\partial t} - \operatorname{div} \left( \rho_n \frac{k_{rn}(S_w)}{\mu_n} \mathbb{K} (\nabla(p_c + p_w) - \rho_n \mathbf{g}) \right) - q_n.
\end{aligned} \tag{0.0.1}$$

For simplicity:  $\rho_w, \rho_n, \mu_w, \mu_n$  constant.

Relative permeabilities

$$k_{rw}(S_w) = S_w^4 \tag{0.0.2}$$

$$k_{rn}(S_w) = (1 - S_w)^2 (1 - S_w^2). \tag{0.0.3}$$

Capillary pressure ( $p_d$  constant)

$$p_c(S_w) = p_d S_w^{-0.5} = p_n - p_w \tag{0.0.4}$$

Jacobian  $J$

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial G_1}{\partial p_w} & \frac{\partial G_1}{\partial S_n} \\ \frac{\partial G_2}{\partial p_w} & \frac{\partial G_2}{\partial S_n} \end{pmatrix} \tag{0.0.5}$$

with  $G_1 = \frac{1}{\rho_w} F_1 + \frac{1}{\rho_n} F_2$  and  $G_2 = F_2$ .

$$\frac{\partial G_1}{\partial p_w} = -\operatorname{div} \left( \left( \frac{k_{rw}(S_w)}{\mu_w} + \frac{k_{rn}(S_w)}{\mu_n} \right) \mathbb{K} \nabla \right) \tag{0.0.6}$$

$$\begin{aligned}
\frac{\partial G_1}{\partial S_n} &= -\operatorname{div} \left( \frac{1}{\mu_w} \frac{dk_{rw}(S_w)}{dS_n} \mathbb{K} (\nabla p_w - \rho_w \mathbf{g}) \right) \\
&\quad - \operatorname{div} \left( \frac{1}{\mu_n} \frac{dk_{rn}(S_w)}{dS_n} \mathbb{K} (\nabla(p_c + p_w) - \rho_n \mathbf{g}) + \frac{1}{\mu_n} k_{rn}(S_w) \mathbb{K} \nabla \frac{dp_c}{dS_n} \right)
\end{aligned} \tag{0.0.7}$$

$$\frac{\partial F_2}{\partial p_w} = -\operatorname{div} \left( \frac{\rho_n}{\mu_n} k_{rn}(S_w) \mathbb{K} \nabla \right) \tag{0.0.8}$$

$$\frac{\partial F_2}{\partial S_n} = -\operatorname{div} \left( \frac{\rho_n}{\mu_n} \frac{dk_{rn}(S_w)}{dS_n} \mathbb{K} (\nabla(p_c + p_w) - \rho_n \mathbf{g}) + \frac{\rho_n}{\mu_n} k_{rn}(S_w) \mathbb{K} \nabla \frac{dp_c}{dS_n} \right) \tag{0.0.9}$$