Based on “Computational Methods For Fluid Dynamics” by Ferziger & Peric (2002)

$$\left(\frac{\partial^2\phi}{\partial x^2}\right)_i \approx \frac{\left(\frac{\partial\phi}{\partial x}\right)_{i+1/2} - \left(\frac{\partial\phi}{\partial x}\right)_{i-1/2}}{0.5\left(x_{i+1}-x_{i-1}\right)} \approx \frac{\frac{\phi_{i+1}+\phi_i}{\Delta x_i} - \frac{\phi_i+\phi_{i-1}}{\Delta x_{i-1}}}{0.5\left(\Delta x_i+\Delta x_{i-1}\right)} = \left[\frac{\phi_{i+1}\Delta x_{i-1}+\phi_{i-1}\Delta x_i}{\Delta x_i+\Delta x_{i-1}} - \phi_i\right]\times\frac{2}{\Delta x_i \Delta x_{i-1}}$$

Let’s find out what is the leading truncation error term.

Firstly, assume Δx_i is the grid size from ‘i’ to ‘i+1’, however, both sides of node point ‘i+0.5Δx_i’ should have the same grid size:

$$\phi_{i+\Delta x_i} = \phi_{i+0.5\Delta x_i} + \left(0.5\Delta x_i\right)\left(\frac{\partial\phi}{\partial x}\right)_{i+0.5\Delta x_i} + \frac{\left(0.5\Delta x_i\right)^2}{2}\left(\frac{\partial^2\phi}{\partial x^2}\right)_{i+0.5\Delta x_i} + \frac{\left(0.5\Delta x_i\right)^3}{6}\left(\frac{\partial^3\phi}{\partial x^3}\right)_{i+0.5\Delta x_i} + \frac{\left(0.5\Delta x_i\right)^4}{24}\left(\frac{\partial^4\phi}{\partial x^4}\right)_{i+0.5\Delta x_i} + O\left(\Delta x_i^{\,5}\right)$$

$$\phi_i = \phi_{i+0.5\Delta x_i} - \left(0.5\Delta x_i\right)\left(\frac{\partial\phi}{\partial x}\right)_{i+0.5\Delta x_i} + \frac{\left(0.5\Delta x_i\right)^2}{2}\left(\frac{\partial^2\phi}{\partial x^2}\right)_{i+0.5\Delta x_i} - \frac{\left(0.5\Delta x_i\right)^3}{6}\left(\frac{\partial^3\phi}{\partial x^3}\right)_{i+0.5\Delta x_i} + \frac{\left(0.5\Delta x_i\right)^4}{24}\left(\frac{\partial^4\phi}{\partial x^4}\right)_{i+0.5\Delta x_i} + O\left(\Delta x_i^{\,5}\right)$$

The upper one minus the lower one,

$$\phi_{i+\Delta x_i} - \phi_i = \left(\Delta x_i\right)\left(\frac{\partial\phi}{\partial x}\right)_{i+0.5\Delta x_i} + \frac{\left(0.5\Delta x_i\right)^3}{3}\left(\frac{\partial^3\phi}{\partial x^3}\right)_{i+0.5\Delta x_i} + O\left(\Delta x_i^{\,4}\right)$$

Therefore, the 1st order derivative with derived with leading truncation error is:

$$\left(\frac{\partial\phi}{\partial x}\right)_{i+0.5\Delta x_i} = \frac{\phi_{i+\Delta x_i} - \phi_i}{\Delta x_i} - \frac{\left(\Delta x_i\right)^2}{24}\left(\frac{\partial^3\phi}{\partial x^3}\right)_{i+0.5\Delta x_i}$$

Similarly,

$$\left(\frac{\partial\phi}{\partial x}\right)_{i-0.5\Delta x_{i-1}} = \frac{\phi_i - \phi_{i-\Delta x_{i-1}}}{\Delta x_{i-1}} - \frac{\left(\Delta x_{i-1}\right)^2}{24}\left(\frac{\partial^3\phi}{\partial x^3}\right)_{i-0.5\Delta x_{i-1}}$$

Now, let's consider

$$\left(\frac{\partial^2\phi}{\partial x^2}\right)_i = \left(\frac{\partial}{\partial x}\frac{\partial\phi}{\partial x}\right)_i = \left(\frac{\partial}{\partial x}\Phi\right)_i \qquad \text{where } \Phi = \frac{\partial\phi}{\partial x}$$

More attention should be paid here since the grid size is different for two sides of node point 'i': left is Δx_{i-1} and right is Δx_i.

$$\Phi_{i+0.5\Delta x_i} = \Phi_i + \left(0.5\Delta x_i\right)\left(\frac{\partial\Phi}{\partial x}\right)_i + \frac{\left(0.5\Delta x_i\right)^2}{2}\left(\frac{\partial^2\Phi}{\partial x^2}\right)_i + \frac{\left(0.5\Delta x_i\right)^3}{6}\left(\frac{\partial^3\Phi}{\partial x^3}\right)_i + \frac{\left(0.5\Delta x_i\right)^4}{24}\left(\frac{\partial^4\Phi}{\partial x^4}\right)_i + O\left(\Delta x_i^{\,5}\right)$$

$$\Phi_{i-0.5\Delta x_{i-1}} = \Phi_i - \left(0.5\Delta x_{i-1}\right)\left(\frac{\partial\Phi}{\partial x}\right)_i + \frac{\left(0.5\Delta x_{i-1}\right)^2}{2}\left(\frac{\partial^2\Phi}{\partial x^2}\right)_i - \frac{\left(0.5\Delta x_{i-1}\right)^3}{6}\left(\frac{\partial^3\Phi}{\partial x^3}\right)_i + \frac{\left(0.5\Delta x_{i-1}\right)^4}{24}\left(\frac{\partial^4\Phi}{\partial x^4}\right)_i + O\left(\Delta x_{i-1}^{\,5}\right)$$

Therefore,

$$\Phi_{i+0.5\Delta x_i} - \Phi_{i-0.5\Delta x_{i-1}} = \left(0.5\Delta x_i + 0.5\Delta x_{i-1}\right)\left(\frac{\partial\Phi}{\partial x}\right)_i + \left[\frac{\left(0.5\Delta x_i\right)^2}{2} - \frac{\left(0.5\Delta x_{i-1}\right)^2}{2}\right]\left(\frac{\partial^2\Phi}{\partial x^2}\right)_i + O\left(\Delta x_i^{\,3}\right)$$

Hence,

$$\left(\frac{\partial\Phi}{\partial x}\right)_i = \frac{\Phi_{i+0.5\Delta x_i} - \Phi_{i-0.5\Delta x_{i-1}}}{0.5\left(\Delta x_i + \Delta x_{i-1}\right)} - \left[\frac{\left(0.5\Delta x_i\right)^2}{2} - \frac{\left(0.5\Delta x_{i-1}\right)^2}{2}\right]\left(\frac{\partial^2\Phi}{\partial x^2}\right)_i + O\left(\Delta x_i^{\,3}\right)$$

Plug in $\Phi = \dfrac{\partial\phi}{\partial x}$, we get

$$\left(\frac{\partial}{\partial x}\frac{\partial \phi}{\partial x}\right)_i = \frac{\left(\frac{\partial \phi}{\partial x}\right)_{i+0.5\Delta x_i} - \left(\frac{\partial \phi}{\partial x}\right)_{i-0.5\Delta x_{i-1}}}{0.5\left(\Delta x_i + \Delta x_{i-1}\right)} - \left[\frac{\left(\Delta x_i\right)^2 - \left(\Delta x_{i-1}\right)^2}{4\left(\Delta x_i + \Delta x_{i-1}\right)}\right]\left(\frac{\partial^2}{\partial x^2}\frac{\partial \phi}{\partial x}\right)_i + O\left(\Delta x_i^{\ 3}\right)$$

$$\approx \frac{1}{0.5\left(\Delta x_i + \Delta x_{i-1}\right)}\left[\frac{\phi_{i+\Delta x_i} - \phi_i}{\Delta x_i} - \frac{\left(\Delta x_i\right)^2}{24}\left(\frac{\partial^3 \phi}{\partial x^3}\right)_{i+0.5\Delta x_i} - \frac{\phi_i - \phi_{i-\Delta x_{i-1}}}{\Delta x_{i-1}} + \frac{\left(\Delta x_{i-1}\right)^2}{24}\left(\frac{\partial^3 \phi}{\partial x^3}\right)_{i-0.5\Delta x_{i-1}}\right] - \left[\frac{\left(\Delta x_i\right)^2 - \left(\Delta x_{i-1}\right)^2}{4\left(\Delta x_i + \Delta x_{i-1}\right)}\right]\left(\frac{\partial^2}{\partial x^2}\frac{\partial \phi}{\partial x}\right)_i$$

Finally,

$$\boxed{\left(\frac{\partial^2 \phi}{\partial x^2}\right)_i = \frac{\frac{\phi_{i+\Delta x_i} - \phi_i}{\Delta x_i} - \frac{\phi_i - \phi_{i-\Delta x_{i-1}}}{\Delta x_{i-1}}}{0.5\left(\Delta x_i + \Delta x_{i-1}\right)} - \frac{1}{0.5\left(\Delta x_i + \Delta x_{i-1}\right)}\left[\frac{\left(\Delta x_i\right)^2}{24}\left(\frac{\partial^3 \phi}{\partial x^3}\right)_{i+0.5\Delta x_i} - \frac{\left(\Delta x_{i-1}\right)^2}{24}\left(\frac{\partial^3 \phi}{\partial x^3}\right)_{i-0.5\Delta x_{i-1}}\right] - \frac{\Delta x_i - \Delta x_{i-1}}{4}\left(\frac{\partial^3 \phi}{\partial x^3}\right)_i}$$

In "Computational Methods For Fluid Dynamics" by Ferziger & Peric (2002), it assumes that:

$$\left(\frac{\partial^3 \phi}{\partial x^3}\right)_{i+0.5\Delta x_i} \approx \left(\frac{\partial^3 \phi}{\partial x^3}\right)_{i-0.5\Delta x_{i-1}} \approx \left(\frac{\partial^3 \phi}{\partial x^3}\right)_i$$

The finally formula becomes:

$$\left(\frac{\partial^2\phi}{\partial x^2}\right)_i = \frac{\dfrac{\phi_{i+\Delta x_i}-\phi_i}{\Delta x_i}-\dfrac{\phi_i-\phi_{i-\Delta x_{i-1}}}{\Delta x_{i-1}}}{0.5\left(\Delta x_i+\Delta x_{i-1}\right)} - \frac{1}{0.5\left(\Delta x_i+\Delta x_{i-1}\right)}\left[\frac{\left(\Delta x_i\right)^2-\left(\Delta x_{i-1}\right)^2}{24}\right]\left(\frac{\partial^3\phi}{\partial x^3}\right)_i - \frac{\Delta x_i-\Delta x_{i-1}}{4}\left(\frac{\partial^3\phi}{\partial x^3}\right)_i$$

$$= \frac{\dfrac{\phi_{i+\Delta x_i}-\phi_i}{\Delta x_i}-\dfrac{\phi_i-\phi_{i-\Delta x_{i-1}}}{\Delta x_{i-1}}}{0.5\left(\Delta x_i+\Delta x_{i-1}\right)} - \frac{\Delta x_i-\Delta x_{i-1}}{12}\left(\frac{\partial^3\phi}{\partial x^3}\right)_i - \frac{\Delta x_i-\Delta x_{i-1}}{4}\left(\frac{\partial^3\phi}{\partial x^3}\right)_i$$

$$= \frac{\dfrac{\phi_{i+\Delta x_i}-\phi_i}{\Delta x_i}-\dfrac{\phi_i-\phi_{i-\Delta x_{i-1}}}{\Delta x_{i-1}}}{0.5\left(\Delta x_i+\Delta x_{i-1}\right)} - \frac{\Delta x_i-\Delta x_{i-1}}{3}\left(\frac{\partial^3\phi}{\partial x^3}\right)_i$$

Which is exactly shown in the book:

$$\left(\frac{\partial^2\phi}{\partial x^2}\right)_i = \frac{\phi_{i+1}(x_i-x_{i-1})+\phi_{i-1}(x_{i+1}-x_i)-\phi_i(x_{i+1}-x_{i-1})}{\frac{1}{2}(x_{i+1}-x_{i-1})(x_{i+1}-x_i)(x_i-x_{i-1})} - \frac{(x_{i+1}-x_i)-(x_i-x_{i-1})}{3}\left(\frac{\partial^3\phi}{\partial x^3}\right)_i + H\ . \tag{3.31}$$