1 Exact Pipe Flow Solutions

We consider numerical computation heat transfer in pipes. Here, we take the simple case of laminar flow, for which exact solutions can be derived in order to provide benchmark tests for more complicated situations. We take as our first test case Hagen-Poiseuille flow with constant heat flux boundary conditions. The Nusselt number, or nondimensional heat transfer coefficient, will be determined by solving the convection-diffusion equation,

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T \quad = \quad \frac{1}{Pe} \nabla^2 T,$$

with thermal boundary conditions

$$\begin{aligned} T(x, y, z = L, t) \quad &= \quad T(x, y, z = 0, t) + \gamma L \text{ (periodicity in } z), \\ \frac{1}{Pe} \nabla T \cdot \hat{\mathbf{n}} \quad &= \quad q'' \text{ on the walls} \end{aligned}$$

For our chosen nondimensionalization, we take the pipe diameter $D = 1$, flux $q'' = 1$, and $\rho C_p = 1$. The axial velocity profile is

$$w(r) \quad = \quad 2\,U\,(1 - 4r^2), \tag{1}$$

where U is the mean velocity. The mean pressure drop[1] for this velocity profile is

$$-\frac{d\bar{p}}{dz} \quad = \quad \frac{32}{Re}.$$

The constant γ arises from the balance between the energy entering via the surface flux and net energy removed by convection. The energy coming in due to the surface flux is $E_{\text{in}} = A_o q''$, where $A_o = \pi D L$ is the outer area of the pipe. The energy extracted by the flow is

$$E_{\text{out}} \quad = \quad \int_{z=L} uT\,dA \; - \; \int_{z=0} uT\,dA \; = \; \int u(\gamma L)\,dA \; = \; \gamma L \int u\,dA \; = \; \gamma L U A_c,$$

where $A_c = \pi D^2/4$ is the cross-sectional area of the pipe. Equating these fluxes we find

$$\gamma \quad = \quad \frac{q'' \pi D L}{L U \pi D^2/4} \; = \; 4,$$

where we have used the fact that $q'' = 1$ and $D = 1$ to arrive at the final value. A more general form for γ that is useful for practical computations is

$$\gamma \quad = \quad \frac{\displaystyle\int_{\partial\Omega_w} q''\,dA}{U \cdot \mathit{volume}},$$

where $\partial\Omega_w$ is the subset of the domain boundary corresponding to the constant thermal-flux surface and *volume* is the domain volume.

For laminar flow under the stated conditions we can find an analytical solution for the temperature. In cylindrical coordinates the fully-developed, steady-state, temperature satisfies

$$-\frac{1}{Pe} \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial T}{\partial r} \quad = \quad -w \frac{\partial T}{\partial z} = -4w,$$

where w is the axial velocity component (1) and $\frac{\partial T}{\partial z} = \gamma = 4$. Integrating twice leads to

$$T(r, z) = 2Pe\left(r^2 \; - \; r^4\right) \; + \; c_1 \ln r \; + \; c_2 + \gamma z.$$

We set $c_1 = 0$ because the solution is bounded at $r = 0$ and $c_2 = 0$ because the solution is determined only up to a constant.

[1] This value is reported as *column 3* in the *volflow* line in the logfile when using Nek5000 with the flow-rate option.

The Nusselt number is the heat transfer coefficient, h, nondimensionalized by the pipe diameter and fluid conductivity,

$$Nu = \frac{Dh}{k}.$$

For our given nondimensionalization we have $D = 1$ and $k = 1/Pe$, which yields $Nu = Pe \cdot h$. The heat transfer coefficient h is defined by the relationship

$$h = \frac{q''}{T_w - T_b},$$

where T_w is the temperature at the wall ($r = 1/2$) and T_b is the bulk (mixing-cup) temperature,

$$T_b := \frac{\int_0^{\frac{1}{2}} w\, T\, 2\pi r\, dr}{\int_0^{\frac{1}{2}} w\, 2\pi r\, dr} = \frac{\int_0^{\frac{1}{2}} w\, T\, 2\pi r\, dr}{A_c U}.$$

In the general case, one must consider temporal and/or spatial averages of T_w and T_b, but the above definitions suffice for the particular case considered here.

Integration of the preceding expressions leads to the bulk temperature

$$T_b = \frac{7Pe}{48} + \gamma z$$

and wall temperature

$$T_w = \frac{3Pe}{8} + \gamma z,$$

from which

$$Nu = Pe \cdot \frac{q''}{T_w - T_b} = \frac{48}{11}.$$